

Zero-Suppressed Computation: A New Computation Inspired by ZDDs*

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Abstract

Zero-suppressed binary decision diagrams (ZDDs) are a data structure representing Boolean functions, and one of the most successful variants of binary decision diagrams (BDDs). On the other hand, BDDs are also called branching programs in computational complexity theory, and have been studied as a computation model. In this paper, we consider ZDDs from the viewpoint of computational complexity theory. Our main proposal of this paper is that we regard the basic idea of ZDDs as a new computation, which we call zero-suppressed computation. We consider the zero-suppressed version of two classical computation models, decision trees and branching programs, and show some results. Although this paper is mainly written from the viewpoint of computational complexity theory, the concept of zero-suppressed computation can be widely applied to various areas.

1 Introduction

Zero-suppressed binary decision diagrams (ZDDs) are a data structure representing Boolean functions, introduced by Minato [6], and one of the most successful variants of binary decision diagrams (BDDs). Knuth has referred to ZDDs as an important variant of BDDs in his book [4], and ZDDs are also referred to in other books [5, 7]. On the other hand, BDDs are also called branching programs in computational complexity theory, and have been studied as a computation model. In this paper, we consider ZDDs from the viewpoint of computational complexity theory.

ZDDs have the same shape as BDDs (and branching programs) have, and the only difference is the way to determine the output. An assignment to the variables determines a computation path from the start node to a sink node. ZDDs output 1 iff the value of the sink node is 1 and all variables which are not contained in the computation path are assigned by 0. (See Section 2 for the formal definitions.) ZDDs have been considered to be effective for Boolean functions whose outputs are almost 0. Our main proposal of

* a preliminary version

this paper is that we regard the basic idea of ZDDs as a new computation, which we call *zero-suppressed computation*. We consider the zero-suppressed version of two classical computation models, decision trees and branching programs.

The first computation model is decision trees. Although decision trees appear in various areas, it is also a computation model to compute Boolean functions in computational complexity theory. We consider zero-suppressed computation for this model. For randomized computation and quantum computation, variants of decision trees (i.e., randomized decision trees and quantum decision trees, respectively) have been well-studied. We define zero-suppressed decision trees and show some gaps of the complexity to deterministic decision trees. Although our results for this model are quite simple observations, it implies a difference between zero-suppressed computation and other computations, and motivates the study of zero-suppressed computation.

The second computation model is branching programs. Branching programs are known as a computation model to approach the L vs. P problem. It is known that the class of decision problems solvable by a nonuniform family of polynomial-size branching programs is equal to $L/poly$ [3]. $L/poly$ is the class of decision problems solvable by nonuniform logarithmic space Turing machines. If one have proven a superpolynomial lower bound for the size of branching programs computing a Boolean function in P , then $L \neq P$. In this paper, we define zero-suppressed branching programs, which actually have the same definition to (unordered) ZDDs, and consider the following question: Is the class of decision problems solvable by a nonuniform family of polynomial-size zero-suppressed branching programs equal to $L/poly$? We prove three results which are related to the question. Firstly, we prove that the class of decision problems solvable by a nonuniform family of polynomial-size width 5 (or arbitrary constant which is greater than 5) zero-suppressed branching programs is equal to nonuniform NC^1 . This corresponds to the well-known Barrington's theorem [1], which showed that the class of decision problems solvable by a nonuniform family of polynomial-size width 5 branching programs is equal to nonuniform NC^1 . Secondly, we prove that the class of decision problems solvable by a nonuniform family of polynomial-size zero-suppressed branching programs contains $L/poly$, and is contained in nonuniform NC^2 . Thirdly, we prove that the class of decision problems solvable by a nonuniform family of polynomial-size read-once zero-suppressed branching programs is equal to the class of decision problems solvable by a nonuniform family of polynomial-size read-once (deterministic) branching programs. When we prove the third result, we also give some insight of the reason why the class of decision problems solvable by a nonuniform family of polynomial-size zero-suppressed branching programs may not be equal to $L/poly$ (Section 4.3).

2 Preliminaries

A Boolean function is a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$.

A (*deterministic*) *branching program* or *binary decision diagram (BDD)* is a directed acyclic graph. The nodes of out-degree 2 are called *inner nodes* and labeled by a variable.

The nodes of out-degree 0 are called *sinks* and labeled by 0 or 1. For each inner node, one of the outgoing edges is labeled by 0 and the other one is labeled by 1. There is a single specific node called the *start node*. An assignment to the variables determines a computation path from the start node to a sink node. The value of the sink node is the output of the branching program or BDD.

A *zero-suppressed binary decision diagram (ZDD)* is also a directed acyclic graph defined in the same way as BDD, and the only difference is the way to determine the output. An assignment to the variables determines a computation path from the start node to a sink node. The ZDD outputs 1 iff the value of the sink node is 1 and all variables which are not contained in the computation path are assigned by 0.

Notice that we define BDD and ZDD with no restriction to the appearance of the variables. (In some papers, BDD and ZDD mean the ordered one, i.e., the variable order is fixed and each variable appears at most once on each path.) The *size* of branching programs is the number of its nodes. If the nodes are arranged into a sequence of levels with edges going only from one level to the next, then the *width* is the size of the largest level. A branching program is called (*syntactic*) *read-once* branching program if each path contains at most one node labeled by each variable.

Decision trees can be defined along the definition of branching programs. We use this way in this paper. A (*deterministic*) *decision tree* is a branching program whose graph is a rooted tree. The start node of a decision tree is the root. We define the (*deterministic*) *decision tree complexity* of f , denoted by $D(f)$, as the depth of an optimal (i.e., minimal-depth) decision tree that computes f .

For a nonnegative integer i , NC^i is the class of decision problems solvable by a uniform family of Boolean circuits with polynomial size, depth $O(\log^i n)$, and fan-in 2.

3 Zero-Suppressed Decision Trees

In this section, we consider decision trees and zero-suppressed computation. We firstly define zero-suppressed decision trees and the zero-suppressed decision tree complexity, and show gaps for the deterministic decision tree complexity.

3.1 Definitions

Since a decision tree is a branching program whose graph is a rooted tree, zero-suppressed decision trees are naturally defined as follows.

A *zero-suppressed decision tree* is also a rooted tree defined in the same way as deterministic decision tree, and the only difference is the way to determine the output. An assignment to the variables determines a computation path from the start node to a sink node. The zero-suppressed decision tree outputs 1 iff the value of the sink node is 1 and all variables which are not contained in the computation path are assigned by 0. We define the *zero-suppressed decision tree complexity* of f , denoted by $Z(f)$, as the depth of an optimal (i.e., minimal-depth) zero-suppressed decision tree that computes f .

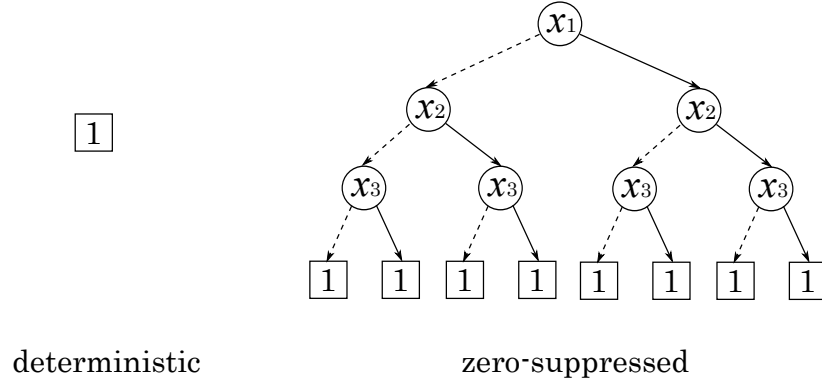


Figure 1: Decision trees computing f for $n = 3$

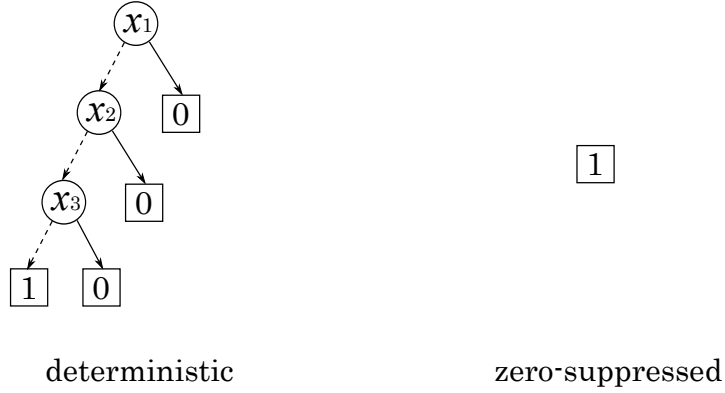


Figure 2: Decision trees computing g for $n = 3$

3.2 Gaps

In the previous subsection, we have defined the zero-suppressed decision tree complexity. We can immediately obtain the following gaps.

Theorem 3.1. *There is a Boolean function f such that $D(f) = 0$ and $Z(f) = n$.*

Proof. Let $f = 1$. (See also Figure 1.) □

Theorem 3.2. *There is a Boolean function g such that $D(g) = n$ and $Z(g) = 0$.*

Proof. Let $g = \neg x_1 \wedge \neg x_2 \wedge \cdots \wedge \neg x_n$. (See also Figure 2.) □

Thus, the advantages and disadvantages of deterministic and zero-suppressed decision trees strongly depend on the Boolean function which decision trees compute. Although these two theorems are quite simple observations, the difference from other computations implies unique behavior of zero-suppressed computation.

Proposition 3.3. $Q_2(f) \leq R_2(f) \leq D(f)$.

$Q_2(f)$ and $R_2(f)$ are variants of decision tree complexity with quantum computation and randomized computation, respectively. For the definitions and the more details, we refer to Section 3 of the survey paper [2].

4 Zero-Suppressed Branching Programs

In this section, we consider branching programs and zero-suppressed computation. Branching programs and BDDs have a same definition as we defined in Section 2. Naturally, we define *zero-suppressed branching programs* as it has the same definition to ZDDs.

4.1 Constant-width zero-suppressed branching programs

Firstly, we prove two lemmas, which are used also in the following subsection.

Lemma 4.1. *Any deterministic branching program of n variables, size s , and width w can be converted to a zero-suppressed branching programs of size $s + n$ and width w .*

Proof. Let G be a deterministic branching program of n variables, size s , and width w . We convert G to a zero-suppressed branching program as follows. We add n nodes, v_1, v_2, \dots, v_n , such that v_i is labeled by x_i for $1 \leq i \leq n$, and connect two outgoing edges of v_i to v_{i+1} for $1 \leq i \leq n - 1$, and connect two outgoing edges of v_n to the 1-sink, and connect all edges which are connected to the 1-sink to v_1 .

In the obtained zero-suppressed branching program, every computation path to the 1-sink contains all variables. Thus, by the definition of zero-suppressed branching programs, G and the obtained zero-suppressed branching program compute the same Boolean function. \square

Lemma 4.2. *Any zero-suppressed branching programs of n variables, polynomial size, and width w can be converted to a Boolean circuit of polynomial size and depth $O(\log w \log n)$.*

Proof. We extend the proof of one direction of the Barrington's theorem. A deterministic branching program of n variables, polynomial size, and width 5 can be converted to a Boolean circuit of polynomial size and depth $O(\log n)$ as follows. Two levels of a deterministic branching program are composed to one level by a circuit of a constant depth. Doing this in parallel and repeating it $O(\log n)$ times yield the desired circuit of depth $O(\log n)$.

If the width is w , two levels of a deterministic branching program are composed to one level by a circuit of depth $O(\log w)$. For the case of zero-suppressed branching programs, we need to memorize the variables contained in the computation path, which can be done with no increase of the depth of the circuit.

Actual encoding of each level is as follows. At most w nodes of each level can be numbered with $\lceil \log w \rceil$ bits. For each outgoing edge of each node of a level, $\lceil \log w \rceil + n$

bits are assigned. The first $\lceil \log w \rceil$ bits represent the node which the outgoing edge connects to. The other n bits represent whether each of n variables is contained in the computation path when the outgoing edge is used in computation. \square

For the case that the width of zero-suppressed branching programs is a constant, we determine that the equivalent class is NC^1 , which is an analog of the Barrington's theorem [1] for deterministic branching programs.

Theorem 4.3. *For any constant $w \geq 5$, the class of decision problems solvable by a nonuniform family of polynomial-size width w zero-suppressed branching programs is equal to nonuniform NC^1 .*

Proof. All problems in nonuniform NC^1 can be solvable by a nonuniform family of polynomial-size width 5 deterministic branching programs [1]. By Lemma 4.1, the problems can be solvable also by a nonuniform family of polynomial-size width 5 zero-suppressed branching programs. Thus, the class contains nonuniform NC^1 .

Consider a problem solvable by a nonuniform family of polynomial-size width w zero-suppressed branching programs. By Lemma 4.2, the problem is also solvable by a nonuniform family of Boolean circuits of polynomial size and depth $O(\log w \log n)$. Since w is a constant, the class is contained in nonuniform NC^1 . \square

4.2 General zero-suppressed branching programs

The main question for zero-suppressed branching programs is whether the class of decision problems solvable by a nonuniform family of polynomial-size zero-suppressed branching programs is equal to L/poly or not. We show a weaker result.

Theorem 4.4. *The class of decision problems solvable by a nonuniform family of polynomial-size zero-suppressed branching programs contains L/poly , and is contained in nonuniform NC^2 .*

Proof. All problems in L/poly can be solvable by a nonuniform family of polynomial-size deterministic branching programs [3]. By Lemma 4.1, the problems can be solvable also by a nonuniform family of polynomial-size zero-suppressed branching programs. Thus, the class contains L/poly .

Consider a problem solvable by a nonuniform family of polynomial-size zero-suppressed branching programs. Obviously, the width of the zero-suppressed branching programs is a polynomial of n . Thus, by Lemma 4.2, the problem is also solvable by a nonuniform family of Boolean circuits of polynomial size and depth $O(\log^2 n)$. Therefore, the class is contained in nonuniform NC^2 . \square

4.3 Read-once zero-suppressed branching programs

In deterministic branching programs, the states in computation are decided only by the node which was reached in computation. Thus, the number of the states is at most the size

of the branching program, and, if the size is at most polynomial, then each state can be represented by logarithmic space, which leads to the fact that the class of decision problems solvable by a nonuniform family of polynomial-size deterministic branching programs is equal to $L/poly$. On the other hand, in zero-suppressed branching programs, the states in computation are not decided only by the node which was reached in computation. It depends on the variables which were contained in the computation path. This is the main reason why the class of decision problems solvable by a nonuniform family of polynomial-size zero-suppressed branching programs may not be equal to $L/poly$. Note that the information of the passed variables cannot be saved in logarithmic space.

In this subsection, we consider a simple case. If deterministic and zero-suppressed branching programs are read-once, then we can convert them to each other with polynomial increase of the size.

Theorem 4.5. *The class of decision problems solvable by a nonuniform family of polynomial-size read-once zero-suppressed branching programs is equal to the class of decision problems solvable by a nonuniform family of polynomial-size read-once deterministic branching programs.*

Proof. We prove two lemmas.

Lemma 4.6. *Any read-once deterministic branching program of n variables and size s can be converted to a read-once zero-suppressed branching program of size $s + 2ns$.*

Proof. Note that the way of the proof of Lemma 4.1 does not give a read-once zero-suppressed branching program. We need more consideration to the place where new nodes are added.

Let G be a read-once deterministic branching program of n variables and size s . Let v_1, v_2, \dots, v_s be the nodes in G such that v_1, v_2, \dots, v_s is a topologically sorted order. We convert G so that every computation path which reaches to a node contains the same all variables, for each node from v_1 to v_s . Assume that every computation path which reaches to v_i contains the same variables for each $1 \leq i \leq k - 1$. We convert G so that every computation path which reaches to v_k contains the same variables as follows. Let X_i be the set of variables which are contained in computation paths to v_i , for $1 \leq i \leq k - 1$. Let X be the union of X_j such that there is an edge from v_j to v_k . Let $X'_i = X - X_i$. For every edge e from v_i to v_k , $1 \leq i \leq k - 1$, we add $|X'_i|$ nodes, $u_1, u_2, \dots, u_{|X'_i|}$, such that the nodes are labeled by the variables contained in X'_i , and connect two outgoing edges of u_j to u_{j+1} for $1 \leq j \leq |X'_i| - 1$, and connect two outgoing edges of $u_{|X'_i|}$ to v_k , and connect e to u_1 . Let G' be the obtained branching program. If computation paths to the 1-sink in G' do not contain all variables, we modify G' to contain all variables by a similar way.

G' is read-once, since added nodes are labeled by the variables contained in X'_i . In G' , every computation path to the 1-sink contains all variables. Thus, by the definition of zero-suppressed branching programs, G and G' compute the same Boolean function. The number of added node is at most n for each edge. \square

Lemma 4.7. *Any read-once zero-suppressed branching program of n variables and size s can be converted to a read-once deterministic branching program of size $s + 2ns$.*

Proof. Let G be a read-once zero-suppressed branching program of n variables and size s . Let v_1, v_2, \dots, v_s be the nodes in G such that v_1, v_2, \dots, v_s is a topologically sorted order. We convert G so that every computation path which reaches to a node contains the same all variables, for each node from v_1 to v_s . Assume that every computation path which reaches to v_i contains the same variables for each $1 \leq i \leq k-1$. We convert G so that every computation path which reaches to v_k contains the same variables as follows. Let X_i be the set of variables which are contained in computation paths to v_i , for $1 \leq i \leq k-1$. Let X be the union of X_j such that there is an edge from v_j to v_k . Let $X'_i = X - X_i$. For every edge e from v_i to v_k , $1 \leq i \leq k-1$, we add $|X'_i|$ nodes, $u_1, u_2, \dots, u_{|X'_i|}$, such that the nodes are labeled by the variables contained in X'_i , and connect the outgoing 0-edge of u_j to u_{j+1} for $1 \leq j \leq |X'_i| - 1$, and connect the outgoing 0-edge of $u_{|X'_i|}$ to v_k , and connect the outgoing 1-edge of u_j to the 0-sink for $1 \leq j \leq |X'_i|$, and connect e to u_1 . Let G' be the obtained branching program. If computation paths to the 1-sink in G' do not contain all variables, we modify G' to contain all variables by a similar way.

G' is read-once, since added nodes are labeled by the variables contained in X'_i . By the definition of zero-suppressed branching programs, G and G' compute the same Boolean function. The number of added node is at most n for each edge. \square

By the two lemmas, the theorem holds. \square

5 Conclusions and Open Problems

In this paper, we proposed zero-suppressed computation as a new computation. For decision trees and branching programs, we could smoothly define the zero-suppressed versions. A challenging open problem is to seek another computation model whose zero-suppressed version is meaningful. When we consider other computation models (e.g., Boolean circuits), it is a difficult and interesting problem even to define the appropriate zero-suppressed version.

Although this paper is mainly written from the viewpoint of computational complexity theory, the concept of zero-suppressed computation can be widely applied to various areas. We show an example. The exactly- k -function $E_k^n(x_1, \dots, x_n)$ is 1 iff $\sum_{i=1}^n x_i = k$. In the standard formulas, an obvious representation of E_1^3 is

$$(x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge \neg x_2 \wedge x_3).$$

In formulas with zero-suppressed computation, E_1^3 is simply represented by

$$(x_1)^z \vee (x_2)^z \vee (x_3)^z,$$

where $()^z$ is a new operation which we define from the concept of zero-suppressed computation and $(f)^z$ is 1 iff $f = 1$ and all variables which are not contained in f are assigned by 0.

For zero-suppressed branching programs, it remains open whether the class of decision problems solvable by a nonuniform family of polynomial-size zero-suppressed branching programs is equal to $L/poly$ or not. We showed some related results to the question in this paper. By Theorem 4.4, there are the following four cases.

- The class is equal to $L/poly$.
- The class is equal to nonuniform NC^2 .
- The class is equal to another known complexity class between $L/poly$ and nonuniform NC^2 .
- The class is not equal to any known complexity class.

Our observation for zero-suppressed decision trees implies unusual properties of zero-suppressed computation, which makes us feel the possibility of some new complexity class.

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